ALL FINITE GENERALIZED TRIANGLE GROUPS

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ABSTRACT. We complete the classification of those generalized triangle groups that are finite.

INTRODUCTION

A generalized triangle group is one given by a presentation $\langle a, b | a^p = b^q = R^m = 1 \rangle$, where p, q, m are integers greater than 1, and R is a word of the form $a^{\alpha_1} b^{\beta_1} \ldots a^{\alpha_k} b^{\beta_k}$, $k \geq 1$, $1 \leq \alpha_i < p$, $1 \leq \beta_i < q$ for $i = 1, \ldots, k$, which is not a proper power. Building on previous results of Baumslag, Morgan and Shalen [1], Conder [2] and Fine, Levin and Rosenberger [3], [4], [6] and [7], Howie, Metaftsis and Thomas [5] were able to give an almost complete list of those generalized triangle groups that are finite.

There are precisely two groups for which they could not decide whether or not they are finite. The two groups in question were

$$G_1 = \langle a, b | a^2 = b^3 = (abababab^2abab^2ab^2)^2 = 1 \rangle$$

and

$$G_2 = \langle a, b | a^2 = b^3 = (abababab^2ab^2abab^2ab^2)^2 = 1 \rangle.$$

Here we show that G_1 is infinite and G_2 is finite of order

$$2^{20} \cdot 3^4 \cdot 5 = 424673280.$$

We are grateful to J. Neubüser and the people at the Lehrstuhl D für Mathematik at the RWTH Aachen for their contributions to this work in the form of helpful conversations. In addition, the first author wants to thank the Fachbereich 1 Mathematik at the RWTH Aachen and in particular J. Neubüser for the hospitality.

2. Theorem. (a) $G_1 = \langle a, b | a^2 = b^3 = (abababab^2abab^2ab^2)^2 = 1 \rangle$ is infinite.

(b) $G_2 = \langle a, b | a^2 = b^3 = (abababab^2ab^2abab^2ab^2)^2 = 1 \rangle$ is finite of order $2^{20} \cdot 3^4 \cdot 5$.

Proof. (a) The commutator subgroup G_1' of G_1 is generated by $x = b$ and $y = aba$, and, in these generators, it has a presentation

$$G_1' = \langle x, y | x^3 = y^3 = xy^{-1}x^{-1}yxyx^{-1}yx^{-1}y^{-1}xyxy^{-1} = 1 \rangle.$$

Let $F \supset \mathbb{Q}$ be an algebraic number field containing an element t with $t^6 - 3t^3 + 1 = 0$. The map

$$x \mapsto X = \begin{pmatrix} A & B & C \\ 0 & 0 & 1 \\ D & E & -A \end{pmatrix},$$

$$y \mapsto Y = \begin{pmatrix} D & E & -A \\ 3(Bt - D) & -D & -Ct \\ 1 & 0 & 0 \end{pmatrix}$$

where $A = -3t^4 + 8t$, $B = -4t^4 + 11t$, $C = 2t^3 - 6$, $D = -5t^5 + 14t^2$ and $E = -7t^5 + 19t^2$ extends to a representation of G_1' over F.

$(XY)^2$ has trace 3 which implies that XY has infinite order. Indeed, if XY would be of finite order then $(XY)^2$, as a matrix of finite order, would be diagonalizable and, hence, having trace 3, it would be the identity matrix I which is not the case. Hence, in particular, G_1 is infinite and ab has infinite order in G_1.

(b) For the computations for the second group G_2 we used the computer system GAP developed at Lehrstuhl D für Mathematik (J. Neubüser) at the RWTH Aachen. G_2 was found to contain a subgroup H of index 24 and generated by a, bab^{-1}, $b^{-1}abab^{-1}ab$, $b^{-1}ab^{-1}ab^{-1}abab^{-1}ababab$, $b^{-1}ab^{-1}ababababab^{-1}ab^{-1}ab^{-1}abab$ and

$$b^{-1}ab^{-1}abab^{-1}ababab^{-1}ab^{-1}abab^{-1}abab.$$

The core N of H has index 13824. To get a presentation for N one constructs the coset table of N using the regular representation of G_2 on H. One then feeds the resulting coset table in the reduced Reidemeister-Schreier algorithm to get a presentation for N in 1208 generators and 5745 relators. This presentation can be reduced via Tietze transformations to a presentation with 7 generators and 148 relators of total length 1584. The latter presentation is manageable with the Todd-Coxeter algorithm to get the order of G_2 which is $2^{20} \cdot 3^4 \cdot 5$. In fact, G_2 is a normal product of $SL(2, 5)$ and a solvable group with amalgamated central subgroup of order 2. □

Remarks. (1) The group G_2 is a counterexample to the conjecture at the end of Chapter I in [6]. Besides this group the conjecture seems to hold.

(2) There is a certain difference between G_1 and G_2 that is quite suggestive. In the case of G_1 there are certain subgroups for which the p-quotient algorithm revealed larger and larger p-factors. On the other hand if one takes G_2 and looks at various subgroups one does not get nilpotent factors of large class, all lower central series stop very soon. This fact was one of the reasons for suspecting that G_2 might be finite.

REFERENCES

1. G. Baumslag, J. W. Morgan and P. B. Shalen, *Generalized triangle groups*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 25–31.

2. M. D. E. Conder, *Three-relator quotients of the modular group*, Quart. J. Math. Oxford (2) **38** (1987), 427–447.

3. B. Fine and G. Rosenberger, *A note on generalized triangle groups*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 233–244.

4. B. Fine, F. Levin and G. Rosenberger, *Free subgroups and decompositions of one-relator products of cyclics. Part* 1: *the Tits alternative*, Arch. Math. **50** (1988), 97–109.

5. J. Howie, V. Metaftsis and R. M. Thomas, *Finite generalized triangle groups*, **347** (1995), 3613–3623.

6. F. Levin and G. Rosenberger, *On free subgroup of generalized triangle groups, Part* II (S. Seghal et al., eds.), Proc. Ohio State meeting in 1992 in honour of H. Zassenhaus (to appear).

7. G. Rosenberger, *On free subgroups of generalized triangle groups*, Algebra i Logika **28** (1989), 227–240.

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